

On Fuzzy Contra e -continuity

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Abstract

In this paper, fuzzy e -open sets and fuzzy e -closed sets are used to define and investigate a new class of functions in fuzzy topological space. Relationships between this new class and other classes of functions are established.

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1 Introduction

Functions and of course continuous functions stand among the most important and most researched points in the whole of the Mathematical Science. Many different forms of continuous functions have been introduced over the years (see [1, 3, 9]). Various interesting problems arise when one consider continuity. Its importance is significant in various areas of mathematics and related sciences. The aim of this paper is to give a new type of fuzzy continuity called fuzzy contra e -continuity. In this connection, two classes of functions were formed namely fuzzy contra pre e -open functions and fuzzy contra pre e -closed functions.

2 Preliminaries

Throughout this paper X , Y and Z are always fuzzy topological spaces. The class of all fuzzy sets on a universe X will be denoted by I^X . Let A be a fuzzy subset of a space X . The fuzzy closure of A , fuzzy interior of A , fuzzy δ -closure

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of A and the fuzzy δ -interior of A are denoted by $cl(A)$, $int(A)$, $cl_\delta(A)$ and $int_\delta(A)$ respectively. A fuzzy subset A of space X is called fuzzy regular open [1] (resp. fuzzy regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The fuzzy δ -interior of fuzzy subset A of X is the union of all fuzzy regular open sets contained in A . A fuzzy subset A is called fuzzy δ -open [8] if $A = int_\delta(A)$. The complement of fuzzy δ -open set is called fuzzy δ -closed (i.e, $A = cl_\delta(A)$).

Definition 2.1. [9] A fuzzy set λ of a fuzzy topological space X is said to be fuzzy e -open if $\lambda \leq cl(int_\delta \lambda) \vee int(cl_\delta \lambda)$, where $cl(\lambda) = \wedge \{ \mu : \mu \geq \lambda, \mu \text{ is fuzzy closed in } X \}$ and $int(\lambda) = \vee \{ \mu : \mu \leq \lambda, \mu \text{ is fuzzy open in } X \}$. If λ is fuzzy e -open, then $1 - \lambda$ is fuzzy e -closed.

Definition 2.2. [9] Let X be a fuzzy topological space and λ be any fuzzy set in X . The fuzzy e -closure of λ in X is denoted by $ecl(\lambda)$ as follows: $ecl(\mu) = \wedge \{ \lambda : \lambda \geq \mu, \lambda \text{ is a fuzzy } e\text{-closed set of } X \}$. Similarly we can define $eint(\lambda)$.

Definition 2.3. [7] A fuzzy set in a fuzzy topological space X is called fuzzy singleton (or fuzzy point) if and only if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is $\alpha (0 < \alpha \leq 1)$ we denote this fuzzy singleton by x_α , where the point x is called its support. For any singleton x_α and any fuzzy set A , we write $x_\alpha \in A$ if and only if $\alpha \leq A(x)$.

Definition 2.4. [2] Let X and Y be two fuzzy topological spaces. Let $\lambda \in I^X$, $\mu \in I^Y$. Then $f(\lambda)$ is a fuzzy subset of Y , defined by $f(\lambda) : Y \rightarrow [1, 0]$

$$f(\lambda)(y) = \begin{cases} \sup_{x \in f^{-1}(\{y\})} \lambda(x) & \text{if } f^{-1}(\{y\}) \neq \phi \\ 0 & \text{if } f^{-1}(\{y\}) = \phi. \end{cases}$$

and $f^{-1}(\mu)$ is a fuzzy subset of X , defined by $f^{-1}(\mu)(x) = \mu(f(x))$.

Definition 2.5. [7] A fuzzy set A in X is said to be q -coincident with a fuzzy set B , denoted by AqB , if there exists $x \in X$ such that $A(x) + B(x) > 1$. It is known that $A \leq B$ if and only if A and $1 - B$ are not q -coincident, denote by $A\bar{q}(1 - B)$.

Definition 2.6. A collection μ of fuzzy sets in a fuzzy space X is said to be cover [5] of a fuzzy set η of X if $(\bigvee_{A \in \mu} A)(x) = 1$, for every $x \in S(\eta)$. A fuzzy cover μ of a fuzzy set η in a fuzzy space X is said to be have a finite subcover if their exists a finite subcollection $\rho = \{A_1, A_2, \dots, A_n\}$ of μ such that $(\bigvee_{j=1}^n A_j)(x) \geq \eta(x)$, for every $x \in s(\eta)$, where $s(\eta)$ denotes the support of a fuzzy set η .

3 Fuzzy contra- e -continuous functions

Definition 3.1. Let X and Y be fuzzy topological space. A function $f : X \rightarrow Y$ is said to be fuzzy contra- e -continuous if for each fuzzy singleton $x_\alpha \in X$ and each fuzzy closed set F in Y containing $f(x_\alpha)$, there exists a fuzzy e -open set U in X containing x_α such that $f(U) \leq F$.

Theorem 3.1. For a function $f : X \rightarrow Y$, the following statements are equivalent:

- (i) f is fuzzy e -continuous;
- (ii) for every fuzzy closed set F in Y , $f^{-1}(F)$ is fuzzy e -open;
- (iii) for every fuzzy open set G in Y , $f^{-1}(G)$ is fuzzy e -closed;
- (iv) for any fuzzy closed set F in Y and for any $x_\alpha \in X$ if $f(x_\alpha) \in F$, then $x_\alpha \in eint(f^{-1}(F))$;
- (v) for any fuzzy closed set F in Y and for any $x_\alpha \in X$ if $f(x_\alpha)qF$, then there exists a fuzzy e -open set U such that $x_\alpha qU$ and $f(U) \leq F$.

Proof.

(i) \Leftrightarrow (ii) Let V be a fuzzy closed set in Y and let $x_\alpha \in f^{-1}(V)$. Since $f(x_\alpha) \in V$, by (i), there exists a fuzzy e -open set U_{x_α} in X containing x_α such that $U_{x_\alpha} \leq f^{-1}(V)$. Thus, $f^{-1}(V)$ is fuzzy e -open. Conversely, let $x_\alpha \in X$ and V be any fuzzy closed set of Y containing $f(x_\alpha)$. By (ii), $f^{-1}(V)$ is fuzzy e -open. If we take $G = f^{-1}(V)$, it follows that $x_\alpha \in G$ and $f(G) \leq V$.

(ii) \Leftrightarrow (iii) Let A be a fuzzy open set in Y . Then, $1 - A$ is fuzzy closed, By (ii), $f^{-1}(1 - A) = 1 - f^{-1}(A)$ is fuzzy e -open. Thus, $f^{-1}(A)$ is fuzzy e -closed.

Converse is similar.

(ii) \Leftrightarrow (iv) Let V be a fuzzy closed set in Y and $f(x_\alpha)qV$. Then, $x_\alpha qf^{-1}(V)$ and from (ii), $f^{-1}(V) \leq eint(f^{-1}(V))$. Hence, $x_\alpha qeint(f^{-1}(V))$.

Converse is obvious.

(iv) \Leftrightarrow (v) Let V be any fuzzy closed set in Y and let $f(x_\alpha)qV$. Then $x_\alpha qeint(f^{-1}(V))$. Take $B = eint(f^{-1}(V))$, then $f(B) = f(eint(f^{-1}(V))) \leq f(f^{-1}(V)) \leq V$.

(v) \Leftrightarrow (iv) Let V be any fuzzy closed set in Y and let $f(x_\alpha)qV$. For V , there exists fuzzy e -open set G such that $x_\alpha qG$ and $f(G) \leq V$. Hence, $G \leq f^{-1}(V)$ and then $x_\alpha qeint(f^{-1}(V))$. ■

Theorem 3.2. Let $f : X \rightarrow Y$ be a function. Then the the following statements are equivalent:

- (i) The function f is fuzzy e -continuous;
- (ii) for each fuzzy singleton x_α of X and each fuzzy open set U in Y with $f(x_\alpha)qU$, there exists a fuzzy e -open set G in X such that $x_\alpha qG$, $f(G) \leq U$.

Proof. (i) \Rightarrow (ii) Let U be any fuzzy open set in Y and let $f(x_\alpha)qU$. By (i), $f^{-1}(U)$ is fuzzy e -open in X . Let $V = f^{-1}(U)$. Then $x_\alpha qU$ and $f(V) \leq U$.

(ii) \Rightarrow (i) Let V be a fuzzy open set in Y and let $x_\alpha qf^{-1}(V)$. Then, $f(x_\alpha)qV$ and thus there exists a fuzzy e -open set U_{x_α} such that $x_\alpha \in U_{x_\alpha}$ and $f(U_{x_\alpha})qV$. Now $x_\alpha qU_{x_\alpha}$ and $U_{x_\alpha} \leq f^{-1}(V)$ and $f^{-1}(V) = \bigcup_{x_\alpha \in f^{-1}(V)} U_{x_\alpha}$. Since union of any fuzzy e -open sets is fuzzy e -open by Theorem 3.7 in [9], $f^{-1}(V)$ is fuzzy e -open in X and hence f is fuzzy e -continuous. ■

Theorem 3.3. If a function $f : X \rightarrow Y$ is fuzzy contra e -continuous and Y is fuzzy regular, then f is fuzzy e -continuous.

Proof. Let x_α be a fuzzy singleton of X and V be a fuzzy open set of Y such that $f(x_\alpha)qV$. Since Y is fuzzy regular, there exists a fuzzy open set W in Y such that $f(x_\alpha)qW$ and $cl(W) \leq V$. Since f is fuzzy contra e -continuous, by Theorem 3.1., there exists a fuzzy e -open set U such that $x_\alpha qU$ and $f(U) \leq cl(W)$. Then $f(U) \leq cl(W) \leq V$. Hence, by Theorem 3.2. f is fuzzy e -continuous. ■

Definition 3.2. A fuzzy topological space X is said to be:

- (i) fuzzy $C-T_1$ [4] (resp. fuzzy $e-T_1$ [9]) if for each pair of distinct fuzzy points x_α and y_β of X there exist fuzzy closed (resp. fuzzy e -open) sets A and B on X such that $x_\alpha \in A$, $y_\beta \in B$, $x_\alpha \notin B$, $y_\beta \notin A$.
- (ii) fuzzy $C-T_2$ [4] (resp. fuzzy $e-T_2$ [9]) if for each pair of distinct fuzzy points x_α and y_β such that $x_\alpha \neq y_\beta$ in X , there exist disjoint fuzzy closed (resp. fuzzy e -open) sets A and B on X such that $x_\alpha \in A$ and $y_\beta \in B$.

Theorem 3.4. If X is a fuzzy topological space and for each pair of disjoint fuzzy singletons x_α and x_β in X , there exist a function f into a fuzzy Urysohn space Y such that $f(x_\alpha) \neq f(x_\beta)$ and f is fuzzy contra contra e -continuous at x_α and y_β , then X is fuzzy $e-T_2$.

Proof. Let x_α and x_β (where $\alpha \neq \beta$) be any disjoint fuzzy singletons of X . Then by hypothesis, there is a fuzzy Urysohn space Y and a function $f : X \rightarrow Y$, which satisfies the conditions of the theorem. Let $y_\theta = f(x_\alpha)$ and $y_\delta = f(x_\beta)$. Then $y_\theta \neq y_\delta$. Since Y is fuzzy Urysohn, there exist fuzzy open sets U_{y_θ} and U_{y_δ} of Y such that $y_\theta q U_{y_\theta}$, $y_\delta q U_{y_\delta}$ and $cl(U_{y_\theta}) \bar{q} cl(U_{y_\delta})$. Since f is fuzzy contra e -continuous at x_α and x_β , there exist a fuzzy e -open sets W_{x_α} and W_{x_β} of X such that $x_\alpha q W_{x_\alpha}$, $x_\beta q W_{x_\beta}$, $f(W_{x_\alpha}) \leq cl(U_{y_\theta})$ and $f(W_{x_\beta}) \leq cl(U_{y_\delta})$. Hence, we get $W_{x_\alpha} \bar{q} W_{x_\beta}$, because $cl(U_{y_\theta}) \bar{q} cl(U_{y_\delta})$. This shows that X is fuzzy $e-T_2$. ■

Corollary 3.1. If $f : X \rightarrow Y$ is a contra- e -continuous injective function, where Y is fuzzy Urysohn, then X is fuzzy $e-T_2$.

Proof. For each pair of disjoint fuzzy singletons x_α and y_β (where $\alpha \neq \beta$) in X , f is a fuzzy contra- e -continuous function of X into a fuzzy Urysohn space Y such that $f(x_\alpha) \neq f(x_\beta)$, because f is injective. Hence by Theorem 3.4., X is fuzzy $e-T_2$. ■

Definition 3.3. A fuzzy topological space X is said to be fuzzy ultra Hausdorff [6] space if for every pair of disjoint singletons of X say x_α and y_β , there exist fuzzy clopen sets, say V_1 and V_2 of X such that $V_1 \bar{q} V_2$.

Theorem 3.5. If $f : X \rightarrow Y$ is a fuzzy contra e -continuous injective function, where Y is fuzzy ultra Hausdorff space, then X is fuzzy $e-T_2$.

Proof. Let x_α and x_β (where $\alpha \neq \beta$) be any two fuzzy singletons of X . Then since f is injective and Y is fuzzy ultra Hausdorff, $f(x_\alpha) \neq f(x_\beta)$ and there exist fuzzy clopen sets, say V_1 and V_2 of Y such that $f(x_\alpha) q V_1$, $f(x_\beta) q V_2$ and $V_1 \bar{q} V_2$. Then $x_\alpha q f^{-1}(V_1)$, $x_\beta q f^{-1}(V_2)$ and $f^{-1}(V_1) \bar{q} f^{-1}(V_2)$. This shows that X is fuzzy $e-T_2$. ■

Theorem 3.6. If $f : X \rightarrow Y$ is fuzzy contra- e -continuous injective function and Y is fuzzy $C-T_1$, then X is fuzzy $e-T_1$.

Proof. Suppose that Y is fuzzy $C-T_1$. For any two fuzzy singletons x_α and y_β in X such that $x_\alpha \bar{q} y_\beta$, there exist fuzzy closed sets A and B of Y such that $f(x_\alpha) q A$, $f(x_\alpha) \bar{q} B$, $f(y_\beta) q B$ and $f(y_\beta) \bar{q} A$. This shows that X is fuzzy $e-T_1$. ■

Theorem 3.7. If $f : X \rightarrow Y$ is fuzzy contra- e -continuous injective function and Y is fuzzy $C-T_2$, then X is fuzzy $e-T_2$.

Proof. Similar to proof of Theorem 3.6. ■

Definition 3.4. A function $f : X \rightarrow Y$ is called fuzzy e -irresolute [10] if the inverse image of each fuzzy e -open set in Y is fuzzy e -open in X .

Theorem 3.8. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Then

- (i) If f is fuzzy e -irresolute and g is fuzzy contra e -continuous, then $g \circ f$ is a fuzzy contra e -continuous function.
- (ii) If f is fuzzy contra e -continuous and g is fuzzy continuous, then $g \circ f$ is a fuzzy contra- e -continuous function.

Proof. Follows from the Definitions. ■

Definition 3.5. A fuzzy topological space X is said to be fuzzy ultra normal [6] if each pair of non-zero fuzzy closed sets V_1 and V_2 such that $V_1 \bar{q} V_2$ can be separated by fuzzy clopen sets F_1 and F_2 such that $F_1 \bar{q} F_2$.

Theorem 3.9. If $f : X \rightarrow Y$ is fuzzy contra- e -continuous closed injective function and Y is fuzzy ultra-normal, then X is fuzzy normal.

Proof. Let F_1 and F_2 be two fuzzy closed subsets of X such that $F_1 \bar{q} F_2$. Since f is closed injective, $f(F_1)$ and $f(F_2)$ are closed subsets of Y such that $f(F_1) \bar{q} f(F_2)$. Since Y is fuzzy ultra-normal $f(F_1)$ and $f(F_2)$ are separated by clopen sets V_1 and V_2 , respectively such that $V_1 \bar{q} V_2$. Hence $F_i \leq f^{-1}(V_i)$, and $f^{-1}(V_i) (i = 1, 2)$ is fuzzy e -open in X and $f^{-1}(V_1) \bar{q} f^{-1}(V_2)$. This shows that, X is fuzzy e -normal. ■

Definition 3.6. A fuzzy filter base Λ is said to be fuzzy e -convergent to a fuzzy singleton x_α in X , if for any fuzzy e -open set U in X such that $x_\alpha q U$, there exists a fuzzy set $V \in \Lambda$ such that $A \leq V$.

Definition 3.7. A fuzzy filter base Λ is said to be fuzzy C -convergent to fuzzy singleton x_α in X , if for any fuzzy closed set V in X such that $x_\alpha q V$, there exists a fuzzy set $A \in \Lambda$ such that $A \leq V$.

Theorem 3.10. If a function $f : X \rightarrow Y$ is fuzzy contra- e -continuous, then for each fuzzy singleton x_α in X and each fuzzy filter base Λ in X e -converging to x_α , the fuzzy filter base $f(\Lambda)$ is fuzzy C -convergent to $f(x_\alpha)$.

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Proof. Let x_α be any fuzzy singleton of X and \wedge be any fuzzy filter base in X e -converging to x_α . Since f is fuzzy contra- e -continuous, then for any fuzzy closed set F in Y such that $Fqf(x_\alpha)$, there exists a fuzzy e -open set U in X such that $x_\alpha qU$ and $f(U) \leq F$. Since \wedge is fuzzy e -converging to x_α , there exists a $B \in \wedge$ such that $B \leq U$. This means that $f(B) \leq F$ and therefore the fuzzy filter base $f(\wedge)$ is fuzzy C -convergent to $f(x_\alpha)$. ■

Theorem 3.11. Let $f : X \rightarrow Y$ be a function and let $g : X \rightarrow X \times Y$ be the fuzzy graph function of f , defined by $g(x_\alpha) = (x_\alpha, f(x_\alpha))$ for every fuzzy singleton x_α in X . If g is fuzzy contra- e -continuous, then f is also fuzzy contra- e -continuous.

Proof. Let A be a fuzzy closed set in Y , then $X \times A$ is a fuzzy closed set in $X \times Y$. Since g is fuzzy contra- e -continuous, then $f^{-1}(A) = g^{-1}(X \times A)$ is fuzzy e -open in X . Thus, f is fuzzy contra- e -continuous. ■

Definition 3.8. A function $f : X \rightarrow Y$ is said to be:

- (i) fuzzy pre e -open if $f(U)$ is fuzzy e -open in Y for every fuzzy e -open set in X .
- (ii) fuzzy pre e -closed if $f(U)$ is fuzzy e -closed in Y for every fuzzy e -closed set in X .
- (iii) fuzzy contra pre e -open if $f(U)$ is fuzzy e -closed in Y for every fuzzy e -open set U of X .
- (iv) fuzzy contra pre e -closed if $f(U)$ is fuzzy e -open in Y for every fuzzy e -closed set U of X .

Remark 3.1.

- (i) Fuzzy contra pre e -openness and fuzzy contra pre e -closedness are equivalent if the function is bijective.
- (ii) Fuzzy contra pre e -openness and fuzzy contra pre e -closedness are independent.

Remark 3.2. For a bijective function $f : X \rightarrow Y$, f is fuzzy contra pre e -open if and only if fuzzy pre e -closed.

The proof of the following Theorems are straightforward and thus omitted.

Theorem 3.12. If $f : X \rightarrow Y$ is a surjective fuzzy pre e -open function and $g : Y \rightarrow Z$ is a function such that $g \circ f : X \rightarrow Z$ is fuzzy contra- e -continuous, then g is fuzzy contra- e -continuous.

Theorem 3.13. Let $f : X \rightarrow Y$ be a surjective fuzzy e -irresolute and fuzzy pre e -open function and let $g : Y \rightarrow Z$ be a function. Then, $g \circ f : X \rightarrow Y$ is fuzzy contra- e -continuous if and only if g is fuzzy contra- e -continuous.

Theorem 3.14. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions such that $g \circ f : X \rightarrow Z$. Then the following properties hold:

- (i) If f is fuzzy pre e -open and g is fuzzy contra pre e -open, then $g \circ f$ is fuzzy contra pre e -open.
- (ii) If f is fuzzy contra pre e -open and g is fuzzy pre e -closed, then $g \circ f$ is fuzzy contra pre e -open.
- (iii) If f is fuzzy pre e -closed and g is fuzzy contra pre e -closed, then $g \circ f$ is fuzzy contra pre e -closed.
- (iv) If f is fuzzy contra pre e -closed and g is fuzzy pre e -open, then $g \circ f$ is fuzzy contra pre e -closed.

Theorem 3.15. For a function $f : X \rightarrow Y$, the following statements are equivalent:

- (i) f is fuzzy contra pre e -open.
- (ii) For every fuzzy subset B of Y and for every fuzzy e -closed subset F of X with $f^{-1}(B) \leq F$, there exists a fuzzy e -open subset O of Y with $B \leq O$ and $f^{-1}(O) \leq F$.
- (iii) For every fuzzy singleton $y_\alpha \leq F$, there exists a fuzzy e -open subset O of Y with $y_\alpha q O$ and $f^{-1}(O) \leq F$.

Proof. (i) \Rightarrow (ii) Let B be any fuzzy subset of Y and let F be a fuzzy e -closed subset of X with $f^{-1}(B) \leq F$. Put $O = 1 - f(1 - f)$. Since f is fuzzy contra pre e -open, then $O = 1 - f(1 - F)$. Since f is fuzzy contra pre e -open, then O is a fuzzy e -open set of Y and since $f^{-1}(B) \leq F$ we have $f(1 - F) \leq (1 - B)$ and hence $B \leq O$. Moreover, $f^{-1}(O) = 1 - (f^{-1}(f(1 - F))) \leq 1 - (1 - F) = F$.

(ii) \Rightarrow (iii) It is sufficient put $B = y_\alpha$.

(iii) \Rightarrow (i) Let A be a fuzzy e -open subset of X . Then let y_α be fuzzy singleton of Y such that $y_\alpha q(1 - f(A))$ and let $F = 1 - A$. By (iii), there exists a fuzzy e -open subset O_{y_α} of V with $y_\alpha q O_{y_\alpha}$ and $f^{-1}(O_{y_\alpha}) \leq F$. Then we see that $y_\alpha q O_{y_\alpha}$ and $O_{y_\alpha} \leq (1 - F)$. Hence $1 - f(A) = \bigcup \{O_{y_\alpha} : y_\alpha q(1 - f(A))\}$ is fuzzy e -open. Therefore, $f(A)$ is a fuzzy e -closed subset in Y and hence f is fuzzy contra pre e -open.

Theorem 3.16. For a function $f : X \rightarrow Y$, the following conditions are equivalent:

- (i) f is fuzzy contra pre e -closed.
- (ii) For every fuzzy subset B of Y and for every fuzzy e -open subset O of X with $f^{-1}(B) \leq O$, there exists a fuzzy e -closed subset F of Y with $B \leq F$ and $f^{-1}(F) \leq O$.

Proof. (i) \Rightarrow (ii) Let B be a fuzzy subset of Y and let O be a fuzzy e -open subset of X with $f^{-1}(B) \leq O$. Put $F = 1 - f(1 - O^c)$. Since f is fuzzy contra pre e -closed, then F is a fuzzy e -closed set of Y and since $f^{-1}(B) \leq O$, we have $f(1 - O^c) \leq B^c$ and hence $B \leq F$. Moreover, $f^{-1}(F) \leq O$.

(ii) \Rightarrow (i) Let E be a fuzzy e -closed subset of X . Put $B = 1 - f(E)$ and let $O = 1 - E$. Hence $f^{-1}(B) = f^{-1}(1 - f(E)) = 1 - (f^{-1}(f(E))) \leq E^c = O$. By assumption there exists a fuzzy e -closed set F of Y for which $B \leq F$ and $f^{-1}(F) \leq O$. It follows that $B = F$. Clearly, if $y_\alpha q F$ and $y_\alpha \bar{q} B$, then $y_\alpha q f(E)$. Therefore, $y_\alpha = f(x_\beta)$ for some singleton x_β of X with $x_\alpha q E$ and we have that $x_\alpha q f^{-1}(F)$ and $f^{-1}(F) \leq O = E^c$, which is a contradiction. Since $B = F$, $f(E)$ is fuzzy e -open in Y and hence f is contra pre e -closed. ■

Theorem 3.17. Let $f : X \rightarrow Y$ be a function. Then,

- (i) If f is fuzzy contra pre e -open, then $ecl(f(O)) \leq (ecl(O))$ for every fuzzy e -open subset O of X .
- (ii) If e is fuzzy contra pre e -closed, then $f(O) \leq eint(f(ecl(O)))$ for every fuzzy subset O of X .

Proof. (i) Since f is fuzzy contra pre e -open, $ecl(f(O)) = f(O) \leq f(ecl(O))$ for every fuzzy e -open set O of X .

(ii) Since f is fuzzy contra pre e -closed and since $ecl(O)$ is fuzzy e -closed, $f(O) \leq f(ecl(O)) = eint(f(ecl(O)))$ for every fuzzy subset O of X . ■

4 Additional properties

Definition 4.1. A fuzzy topological space X is said to be fuzzy connected (resp. fuzzy e -connected [9]) if there does not exist fuzzy open (resp. fuzzy e -open) sets A and B such that $A \vee B = 1$, $A \neq 0$, $B \neq 0$.

Theorem 4.1. If $f : X \rightarrow Y$ is a fuzzy contra- e -continuous surjective function and X is fuzzy e -connected, then Y is fuzzy connected.

Proof. Suppose that Y is not a fuzzy connected space. Then there exist fuzzy open sets A_1 and A_2 such that $A_1 \vee A_2 = 1$, $A_1 \neq 0$, $A_2 \neq 0$. Therefore, A_1 and A_2 are fuzzy clopen sets. Since f is fuzzy contra- e -continuous, $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are fuzzy e -open sets in X . Moreover, $f^{-1}(A_1) \vee f^{-1}(A_2) = 1$, $f^{-1}(A_1) \neq 0$, $f^{-1}(A_2) \neq 0$. This shows that X is not fuzzy e -connected, which is a contradiction to the assumption that X is fuzzy connected. By contradiction, Y is fuzzy connected. ■

Definition 4.2. A fuzzy topological space X is called:

- (i) fuzzy e -ultra-connected if every two nonzero e -closed subsets of X say A_1 and A_2 is $A_1 q A_2$.
- (ii) fuzzy hyperconnected if every fuzzy open set is dense.

Theorem 4.2. If $f : X \rightarrow Y$ is fuzzy contra- e -continuous surjective function, where X is a fuzzy e -ultra-connected space, then Y is fuzzy hyperconnected.

Proof. Assume that Y is not fuzzy hyperconnected. Then there exists a fuzzy open set A such that A is not dense in Y . Then there exist fuzzy open sets A_1 and A_2 , where $A_1 \neq 0$, $A_2 \neq 0$ and $A_1 \bar{q} A_2$ in Y . Set $A_1 = \text{int}(cl(B_1))$ and $A_2 = 1 - cl(B_1)$. Since f is fuzzy contra- e -continuous surjective, by Theorem 3.1., $C_1 = f^{-1}(A_1)$ and $C_2 = f^{-1}(A_2)$ are nonzero fuzzy e -closed sets in X such that $f^{-1}(A_1) \bar{q} f^{-1}(A_2)$. By assumption, the fuzzy e -ultra-connectedness of X implies that $C_1 q C_2$. By contradiction, Y is fuzzy hyperconnected. ■

Definition 4.3. A collection μ of fuzzy sets in a fuzzy space X is said to be cover [5] of a fuzzy set η of X if $(\bigcup_{A \in \mu} A)(x) = 1$, for every $x \in S(\eta)$. A fuzzy cover μ of a fuzzy set η in a fuzzy space X is said to have a finite subcover if there exists a finite subcollection $\rho = \{A_1, A_2, \dots, A_n\}$ of μ such that $(\bigvee_{j=1}^n A_j)(x) \geq \eta(x)$, for every $x \in s(\eta)$, where $s(\eta)$ denotes the support of a fuzzy set η .

Definition 4.4. A fuzzy topological space X is said to be:

- (i) fuzzy e -closed-compact (resp. fuzzy compact) if every fuzzy e -closed (fuzzy closed) cover of X has a finite subcover.
- (ii) fuzzy countably e -closed (resp. fuzzy countably closed) if every countable subcover of X by fuzzy e -closed (resp. fuzzy open) sets has a finite subcover.
- (iii) fuzzy e -closed-Lindelöf (resp. fuzzy Lindelöf) if every cover of X by fuzzy e -closed (resp. fuzzy open) sets has a countable subcover.

Theorem 4.3. Let $f : X \rightarrow Y$ be fuzzy contra- e -continuous surjective function. Then the following statements hold:

- (i) If X is fuzzy e -closed-compact, then Y is fuzzy compact.
- (ii) If X is fuzzy countably e -closed, then Y is fuzzy countably closed.
- (iii) If X is fuzzy e -closed-Lindelöf, then Y is fuzzy Lindelöf.

Proof. (i) Let $\{A_\alpha : \alpha \in I\}$ be any fuzzy open cover of Y . Since f is fuzzy contra- e -continuous, then we have $\{f^{-1}(A_\alpha) : \alpha \in I\}$ is fuzzy e -closed cover of X . Since X is fuzzy e -closed-compact, there exists a finite subset I_0 of I such that $\bigvee\{f^{-1}(A_\alpha) : \alpha \in I_0\} = 1$. Thus, we have $\{A_\alpha : \alpha \in I_0\} = 1$ and Y is fuzzy compact. The other proofs are similarly. ■

Definition 4.5. A fuzzy topological space X is said to be:

- (i) fuzzy e -compact [10] (resp. fuzzy strongly S -closed [4]) if every fuzzy e -open (resp. fuzzy closed) cover of X had a finite subcover.
- (ii) fuzzy countably e -compact (resp. fuzzy strong countably S -closed [4]) if every countable cover of X by fuzzy e -open (resp. fuzzy closed) sets has a finite subcover.
- (iii) fuzzy e -Lindelöf (resp. fuzzy strongly S -Lindelöf) if every fuzzy e -open (resp. fuzzy closed) cover of X has a countable subcover.

Theorem 4.4. Let $f : X \rightarrow Y$ be a fuzzy contra- e -continuous surjective function. Then the following statements hold:

- (i) If X is fuzzy e -compact, then Y is fuzzy strongly S -closed.

- (ii) If X is fuzzy e -Lindelöf, then Y is fuzzy strongly S -Lindelöf.
- (iii) If X is fuzzy countably e -compact, then Y is fuzzy strong countable S -closed.

Proof. Similar proof of Theorem 4.3. ■

Recall that for a function $f : X \rightarrow Y$, the fuzzy subset $\{(x_\alpha, f(x_\alpha)) : x_\alpha \in X\} \subseteq X \times Y$ is called the fuzzy graph of f and is denoted by $G(f)$.

Definition 4.6. The fuzzy graph $G(f)$ of a function $f : X \rightarrow Y$ is said to be fuzzy contra- e -closed if for each $(x_\alpha, y_\beta) \in (X \times Y) \setminus G(f)$, there exist a fuzzy e -open set A in X containing x_α and a fuzzy closed set B in X containing y_β such that $(A \times B) \wedge G(f) = 0$.

Lemma 4.1. The following properties are equivalent for the fuzzy graph $G(f)$ of a function f :

- (i) $G(f)$ is fuzzy contra- e -closed.
- (ii) for each $(x_\alpha, y_\beta) \in (X \times Y) \setminus G(f)$, there exist a fuzzy e -open set A in X containing x_α and a fuzzy closed set B containing y_β such that $f(A) \wedge B = 0$.

Theorem 4.5. If $f : X \rightarrow Y$ is fuzzy contra- e -continuous and Y is fuzzy Urysohn, $G(f)$ is fuzzy contra- e -closed in $X \times Y$.

Proof. Suppose that Y is fuzzy Urysohn. Let $(x_\alpha, y_\beta) \in (X \times Y) \setminus G(f)$. It follows that $f(x_\alpha) \neq y_\beta$. Since Y is fuzzy Urysohn, there exist fuzzy open sets A and B such that $f(x_\alpha) \in A$, $y_\beta \in B$ and $cl(A) \wedge cl(B) = 0$. Since f is fuzzy contra- e -continuous, there exists a fuzzy e -open set C in X containing x_α such that $f(C) \leq cl(A)$. Therefore, $f(C) \wedge cl(B) = 0$ and $G(f)$ is fuzzy contra- e -closed in $X \times Y$. ■

Theorem 4.6. Let $f : X \rightarrow Y$ have a fuzzy contra- e -closed graph. If f is injective, the X is fuzzy e - T_1 .

Proof. Let x_α and y_β be any two distinct fuzzy singletons in X . Then, we have $(x_\alpha, f(y_\beta)) \in (X \times Y) \setminus G(f)$. By Lemma 4.1., there exist a fuzzy e -open set A in X containing x_α and a fuzzy closed set B in Y containing $f(y_\beta)$ such that $f(A) \wedge B = 0$; hence $A \wedge f^{-1}(B) = 0$. Therefore, we have $y_\beta \notin A$. This implies that X is fuzzy e - T_1 . ■

Theorem 4.7. If $f : X \rightarrow Y$ is fuzzy contra- e -continuous injective function and Y is fuzzy Urysohn, then X is fuzzy $e-T_2$

Proof. Suppose that Y is fuzzy Urysohn. By the injective of f , it follows that $f(x_\alpha) \neq f(y_\beta)$ for any distinct fuzzy singletons x_α and y_β in X . Since Y is fuzzy Urysohn, there exist fuzzy open sets A and B such that $f(x_\alpha) \in A$, $f(y_\beta) \in B$ and $cl(A) \wedge cl(B) = 0$. Since f is fuzzy contra- e -continuous, there exist fuzzy e -open sets C and D in X containing x_α and y_β , respectively, such that $f(C) \leq cl(A)$ and $f(D) \leq cl(B)$. Hence $C \wedge D = 0$. This shown that X is fuzzy $e-T_2$. ■

Theorem 4.8. Let (X_i, τ_i) be a fuzzy topological space for all $i \in I$ and I be finite. Suppose that $(\prod_{i \in I} X_i, \sigma)$ is a product space and $f : (X, \tau) \rightarrow (\prod_{i \in I} X_i, \sigma)$ is any function. If f is fuzzy contra- e -continuous, then $pr_i \circ f$ is fuzzy contra- e -continuous where pr_i is projection function for each $i \in I$.

Proof. Let $x_\alpha \in X$ and $(pr_i \circ f)(x_\alpha) \in A_i$ and A_i be a fuzzy closed set in (X_i, τ_i) . Then $f(x_\alpha) \in pr_i^{-1}(A_i) = A_i \times \prod_{j \neq i} X_j$ a fuzzy closed set in $(\prod_{i \in I} X_i, \sigma)$. Since f is fuzzy contra- e -continuous, there exists a fuzzy e -open set B containing x_α such that $f(B) \leq A_i \times \prod_{j \neq i} X_j = pr_i^{-1}(A_i)$ and hence $B \leq (pr_i \circ f)^{-1}(A_i)$ and we obtain that $pr_i \circ f$ is fuzzy contra- e -continuous for each $i \in I$. ■

References

- [1] K. K. Azad, *On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity*, J. Math. Anal. Appl., **82** (1981) 14-32.
- [2] C. L. Chang, *Fuzzy topological space*, J. Math. Anal. Appl., **24** (1968), 182-190.
- [3] E. Ekici, *On fuzzy functions*, Commun. Korean. Math. Soc., **20** (2005), 4, 781-789.
- [4] E. Ekici and E. Kerre, *On fuzzy contra-continuities*, Adv. Fuzzy Math., **1** (1) (2006), 35-44.
- [5] S. Ganguly and S. Saha, *A note on compactness in fuzzy setting*, Fuzzy Sets and Systems, **34** (1990), 117-124.
- [6] N. Gowrisankar, *Fuzzy totally γ -continuous functions*, The Journal of Fuzzy Mathematics, **18** (4), 2010.
- [7] P. Pao-Ming and L. Ying-Ming, *Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-smith convergence*, J. Math. Anal. Appl., **76** (1980), 571-599.

- [8] A. Mukherjee and S. Debnath, *δ -semi-open sets in fuzzy setting*, J. Tri. Math. Soc., **8** (2006) 51-54.
- [9] V. Seenivasan and K. Kamala, *Fuzzy e -continuity and fuzzy e -open sets*, Annals of Fuzzy Mathematics and Informatics, **8** (1) (2014), 141-148.
- [10] R. V. M. Rangarajan, M. Chandrasekharan, A. Vadivel and M. Palanisamy, *Fuzzy e -irresolute mappings on fuzzy topological spaces*, Annals of Fuzzy Mathematics and Informatics, **11** (1), (2016), 53-62.